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The vibration and stability of orthotropic conical shells with non-homogeneous material properties under a hydrostatic pressure

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Abstract

In this paper, the vibration and stability of orthotropic conical shells with non-homogeneous material properties under a hydrostatic pressure are studied. At first, the basic relations have been obtained for orthotropic truncated conical shells, Young's moduli and density of which vary continuously in the thickness direction. By applying the Galerkin method to the foregoing equations, the buckling pressure and frequency parameter of truncated conical shells are obtained from these equations. Finally, carrying out some computations, the effects of the variations of conical shell characteristics, the effects of the non-homogeneity and the orthotropy on the critical dimensionless hydrostatic pressure and lowest dimensionless frequency parameter have been studied, when Young's moduli and density vary together and separately. The results are presented in tables, figures and compared with other works.

1. Introduction

In accordance with the recent developments of science and technology, structural elements composed of non-homogeneous materials are becoming increasingly used in aerospace industry (e.g., reactor caps, some critical parts of bullets and rockets, space vehicles, pipes, nuclear reactors) and other high technology problems due to their high specific strength and specific stiffnesses, lightweight, resistance fatigue and better design behavior. Materials and structural components are often non-homogeneous, either by design or due to the physical composition and imperfections in the underlying materials. However, the non-homogeneity of materials stems from production techniques, surface and thermal polishing processes, effect of radiation, etc. Thus, the physical properties of materials change continuously from point to point as continuous functions of coordinates, e.g., the computation of structural members exposed to radiation effects is simplified by considering a variation of physical properties in the thickness direction, as an initial approximation.

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Nomenclature

E_0	Young's modulus of the homogeneous isotropic conical shell
E_{0S}, $E_{0\theta}$	Young's moduli of the homogeneous orthotropic conical shells
E_S, E_θ	Young's moduli of the non-homogeneous orthotropic conical shell
$f(t)$	time-dependent amplitude
h	thickness of the conical shell
L	length of the conical shell
L_1	length of the cylindrical shell
M_S, M_θ, $M_{S\theta}$	moment resultants
N_S, N_θ, $N_{S\theta}$	force resultants
$N_s^0, N_\theta^0, N_{S\theta}^0$	membrane forces prior to buckling
n, m	wavenumbers
n_{cr}, n_{1cr}	circumferential wavenumbers corresponding to critical dimensionless hydrostatic pressure and lowest dimensionless frequency parameter, respectively
P	hydrostatic pressure
P_{cr}, P_{1cr}	critical hydrostatic pressure and critical dimensionless hydrostatic pressure, respectively
q	power-law index
Q_{ij}	reduced stiffness
R_1, R_2	average radii of the small and large bases of the conical shell
R	radius of the cylindrical shell
S	the coordinate axis through the vertex on the reference surface of the cone
S_1, S_2	the inclined distances of the bases of the cone from the vertex, respectively
w	displacement of the reference surface in the inwards normal direction z
z	independent variable
$Z_B, \overline{Z}_B$	Batdorf conical and cylindrical shells parameters, respectively

Greek letters

γ	semi-vertex angle of the cone
$\varepsilon_S^0, \varepsilon_\theta^0, \varepsilon_{S\theta}^0$	strain components on the reference surface of the conical shell
ζ	the coordinate axis in the inwards normal direction of the reference surface of the cone
$\overline{\zeta} = \zeta/h$	dimensionless thickness coordinate
θ	angle of rotation around the longitudinal axis starting from a radial plane
μ	non-homogeneity coefficient
ν_0	Poisson's ratio of the homogeneous isotropic conical shell
$\nu_{\theta S}$, $\nu_{S\theta}$	Poisson's ratios of the orthotropic conical shell
ρ_0, ρ	densities of the homogeneous and non-homogeneous orthotropic conical shells, respectively
σ_S, σ_θ, $\sigma_{S\theta}$	stress components
υ	non-homogeneity parameter
$\varphi_j(\overline{\zeta})$ $(j = 1, 2)$	continuous functions of the non-homogeneity
Ψ	stress function
ω	frequency
ω_1	dimensionless frequency parameter
$\overline{\varpi}$	cyclic natural frequency

Important contributions to the elasticity theory of non-homogeneous materials and structures have been brought [1–4].

Thereafter, notable contributions were made [5–9] dealing with various types of non-homogeneity considerations. In most of these researches, the variation of the elasticity modulus was assumed to be unbounded and exponential or power functions of the radial or axial coordinates were used. In all the above studies, Poisson's ratio was kept constant while the non-homogeneity parameter (or non-homogeneity function) was assumed arbitrary.

The studies of non-homogeneous orthotropic plates and cylindrical shells with uniform/non-uniform thickness with various edge conditions have been carried out after 1985 by a number of researchers such as Refs. [10–17] just to mention the prominent ones. Similarly in three recent studies [18–20], the work dealing with free vibrations of inhomogeneous membranes has been reported. There is no restriction for non-homogeneity parameter or non-homogeneity function in all studies mentioned above. However, in actual engineering applications, the variation of the elastic properties of materials remains in a bounded range and small enough, necessitating a restriction on the variation (non-homogeneity) functions. Researchers [21–24] made some restrictions, by assuming the non-homogeneity functions continuous and less than unity.

Conical shells have been widely used as important structural components in practical engineering applications, and the vibration and buckling analyses of such components is important for the overall safety and stability of the whole structure. The buckling behavior of conical shells under hydrostatic pressure was first studied by Niordson, and a number of investigations have been made on mechanical behavior isotropic shells [25–35]. For orthotropic shells, there have been fewer studies. Singer [36,37] derived a set of equations for the buckling of orthotropic conical shells by using the displacement–strain relations given in Ref. [26]. Serpico [38] extended the one-term Reyleigh–Ritz approach of Niordson [24] to the orthotropic conical shells; but the results are only reliable for short cones. Singer and Fershst-Scher [39] obtained solutions for the buckling of orthotropic conical shells under hydrostatic pressure. Refs. [40–43] studied the vibration and the buckling problems of the homogeneous orthotropic conical shells.

The equations of motion for conical shells, according to various thin shell theories, exist in Leissa [44]. The free vibration of isotropic conical shells has been widely studied by many researchers using various methods [45–53]. In addition to the work done for free vibrations of isotropic cones, there exists less studies for free vibration of orthotropic conical shells [54–62].

Vibration and buckling of a general conical shell depends on various parameters, such as the geometric properties of the shell (the cone semi-vertex angle, the radius at one end, the slant length of the shell and the thickness), the material properties (isotropic, orthotropic, composite, laminated, etc.) and the type of the applied load (axial compression, hydrostatic pressure, torsion and combined load).

For a more realistic vibration and buckling analyses of conical shells, the influence of the non-homogeneity of the material must be considered as well as the above-mentioned factors. When the conical shell is made of non-homogeneous orthotropic material, solution of the vibration and stability problems becomes more complicated. The literature on the stability problems of non-homogeneous truncated conical shells under lateral pressure is very scarce [63–65]. Since the free vibration and the stability of non-homogeneous orthotropic truncated conical shells under hydrostatic pressure have not been studied yet, the objective of the present research is to investigate the vibration and stability of orthotropic composite truncated conical shells subjected hydrostatic pressure, when Young's moduli and density vary continuously through the thickness coordinate direction.

2. Basic equations

Consider a circular non-homogeneous orthotropic truncated conical shell as shown in Fig. 1, where γ is the semi-vertex cone angle, L the length, h the thickness, R_1 and R_2 the radii at the ends. The reference surface of the conical shell is taken as the middle surface where an orthogonal coordinate system (S, θ, ζ) is fixed. The S-axis lies on the curvilinear middle surface of the cone, S_0 and S_1 being the coordinates of the points where this axis intersects the small and large bases, respectively. Furthermore, the ζ-axis is always normal to the moving S-axis, lying in the plane generated by the S-axis and the axis of the cone, and points inwards. The θ-axis is in the direction perpendicular to the $S-\zeta$ plane. The axes of orthotropy are parallel to the curvilinear

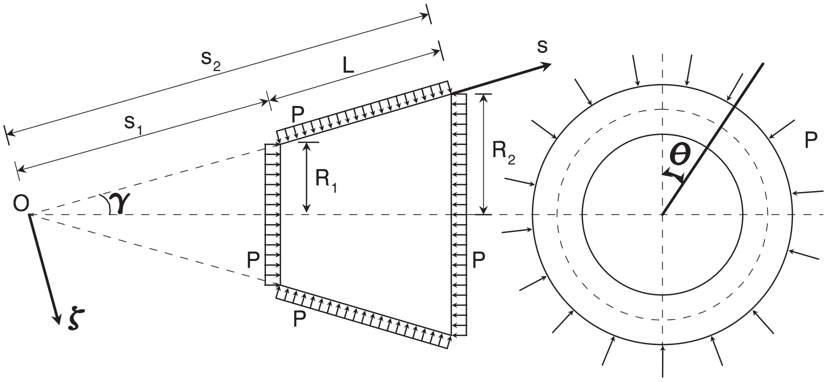

Fig. 1. Geometry of a truncated conical shell.

coordinates S and θ. The non-homogeneity of the material of the shell is assumed to arise due to the variation of Young's moduli and density along the thickness direction ζ as [1,2,18]

$$[E_S(\bar{\zeta}), E_\theta(\bar{\zeta}), G_0(\bar{\zeta})] = \bar{\varphi}_1(\bar{\zeta})[E_{0S}, E_{0\theta}, G_0], \quad \rho(\bar{\zeta}) = \rho_0\bar{\varphi}_2(\bar{\zeta}), \quad \bar{\zeta} = \zeta/h \tag{1}$$

where E_{0S} and $E_{0\theta}$ are Young's moduli in the S and θ directions, respectively. G_0 is the shear modulus on the plane and ρ_0 the density of homogeneous orthotropic material of shell. Additionally,

$$\bar{\varphi}_j(\bar{\zeta}) = 1 + \mu\varphi_j(\bar{\zeta}), \quad j = 1,2 \tag{2}$$

where $\varphi_1(\bar{\zeta})$ and $\varphi_2(\bar{\zeta})$ are continuous functions of non-homogeneity defining the variations of Young's moduli and density, respectively, satisfying the condition $|\varphi_j(\bar{\zeta})| \leqslant 1$, and μ is a non-homogeneity coefficient, satisfying $0 \leqslant \mu < 1$. The non-homogeneity functions of the material of the truncated conical shell are assumed to be power and exponential functions which [1–8,14,18,65]

$$\varphi_j(\bar{\zeta}) = \pm\bar{\zeta}^q, \quad j = 1,2, \quad q = 1,2,3\ldots \tag{3}$$

$$\varphi_j(\bar{\zeta}) = \pm e^{-0.1|\bar{\zeta}|}\cos(\upsilon\bar{\zeta}), \quad j = 1,2 \tag{4}$$

where q is the power-law index and υ is a real number and it is called non-homogeneity parameter or frequency of the non-homogeneity in direction $O\zeta$.

The stress–strain relation for thin non-homogeneous orthotropic truncated conical shells is

$$\begin{pmatrix} \sigma_S \\ \sigma_\theta \\ \sigma_{S\theta} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_S^0 - \zeta\dfrac{\partial^2 w}{\partial S^2} \\ \varepsilon_\theta^0 - \zeta\left(\dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S}\right) \\ \varepsilon_{S\theta}^0 - \zeta\left(\dfrac{1}{S}\dfrac{\partial^2 w}{\partial S\,\partial\theta_1} - \dfrac{1}{S^2}\dfrac{\partial w}{\partial\theta_1}\right) \end{bmatrix} \tag{5}$$

where σ_S, σ_θ and $\sigma_{S\theta}$ are the stresses, ε_S^0 and ε_θ^0 are the normal strains in the curvilinear coordinate directions S and θ on the middle surface, respectively, while $\varepsilon_{S\theta}^0$ is the corresponding shear strain, $\theta_1 = \theta\sin\gamma$, w is the displacement of the middle surface in the normal direction, positive towards the axis of the cone and assumed to be much smaller than the thickness. The quantities Q_{ij}, $i, j =$ 1,2,6 for an orthotropic lamina are

$$Q_{11} = \frac{E_{0S}\bar{\varphi}_1(\bar{\zeta})}{1 - \nu_{S\theta}\nu_{\theta S}}, \quad Q_{22} = \frac{E_{0\theta}\bar{\varphi}_1(\bar{\zeta})}{1 - \nu_{S\theta}\nu_{\theta S}}, \quad Q_{12} = \nu_{\theta S}Q_{11}, \quad Q_{21} = \nu_{S\theta}Q_{22}, \quad Q_{66} = 2G_0\bar{\varphi}_1(\bar{\zeta}) \tag{6}$$

in which $\nu_{S\theta}$ and $\nu_{\theta S}$ are Poisson's ratios, assumed to be constant and satisfying $\nu_{\theta S}E_{0S} = \nu_{S\theta}E_{0\theta}$ [66,67].

The well-known force and moment resultants are expressed by [33,35]

$$[(N_S, N_\theta, N_{S\theta}), ((M_S, M_\theta, M_{S\theta})] = \int_{-h/2}^{h/2} (1,\zeta)(\sigma_S, \sigma_\theta, \sigma_{S\theta})\,d\zeta \tag{7}$$

The relations between the forces N_S, N_θ and $N_{S\theta}$ and the stress function Ψ are given by

$$(N_S, N_\theta, N_{S\theta}) = \left(\frac{1}{S^2}\frac{\partial^2\Psi}{\partial\theta_1^2} + \frac{1}{S}\frac{\partial\Psi}{\partial S}, \frac{\partial^2\Psi}{\partial S^2}, -\frac{1}{S}\frac{\partial^2\Psi}{\partial S\,\partial\theta_1} + \frac{1}{S^2}\frac{\partial\Psi}{\partial\theta_1}\right) \tag{8}$$

The orthotropic truncated conical shell is simply supported and subjected to a hydrostatic pressure [29]:

$$N_S^0 = -0.5PS\tan\gamma, \quad N_\theta^0 = -PS\tan\gamma, \quad N_{S\theta}^0 = 0 \tag{9}$$

where N_s^0, N_θ^0 and $N_{S\theta}^0$ are the membrane forces for the condition with zero initial moments. Substituting expressions (5) in Eq. (7) after some rearrangements, the relations found for moments and strains, being substituted in Donnell-type stability and compatibility equations of truncated conical shells [33] together with relation (8), then considering the independent variables $S = S_1e^z$ and $\Psi = \Psi_1e^{2z}$, after lengthy computations, the system of differential equations for w and Ψ_1 can be obtained as

$$\begin{aligned}
\Theta \equiv\; & c_{12}e^{2z}\frac{\partial^4\Psi_1}{\partial z^4} + (c_{11} - c_{22} + 4c_{12})e^{2z}\frac{\partial^3\Psi_1}{\partial z^3} \\
& + (5c_{12} + 3c_{11} - 3c_{22} - c_{21} + S_1e^z \cot\gamma)e^{2z}\frac{\partial^2\Psi_1}{\partial z^2} \\
& + (2c_{11} - 2c_{22} + 2c_{12} - 2c_{21} + 3S_1e^z \cot\gamma)e^{2z}\frac{\partial\Psi_1}{\partial z} + 2S_1e^{3z}\Psi_1 \cot\gamma \\
& + c_{21}e^{2z}\frac{\partial^4\Psi_1}{\partial\theta_1^4} + (c_{11} - 2c_{31} + c_{22})e^{2z}\frac{\partial^4\Psi_1}{\partial z^2\,\partial\theta_1^2} + (c_{11} - 4c_{31} + 3c_{22})e^{2z}\frac{\partial^3\Psi_1}{\partial z\,\partial\theta_1^2} \\
& + 2(c_{22} - c_{31} + c_{21})e^{2z}\frac{\partial^2\Psi_1}{\partial\theta_1^2} - c_{24}\frac{\partial^4 w}{\partial\theta_1^4} - (c_{14} + c_{23} + 2c_{32})\frac{\partial^4 w}{\partial z^2\,\partial\theta_1^2} \\
& + (3c_{14} + c_{23} + 4c_{32})\frac{\partial^3 w}{\partial z\,\partial\theta_1^2} - 2(c_{14} + c_{32} + c_{24})\frac{\partial^2 w}{\partial\theta_1^2} - c_{13}\frac{\partial^4 w}{\partial z^4} \\
& + (4c_{13} + c_{23} - c_{14})\frac{\partial^3 w}{\partial z^3} - (5c_{13} + 3c_{23} - 3c_{14} - c_{24})\frac{\partial^2 w}{\partial z^2} + 2(c_{13} + c_{23} - c_{14} - c_{24})\frac{\partial w}{\partial z} \\
& - \frac{PS_1^3e^{3z}\tan\gamma}{2}\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial w}{\partial z} + 2\frac{\partial^2 w}{\partial\varphi^2}\right) - S_1^4e^{4z}\rho_t h\frac{\partial^2 w}{\partial t^2} = 0
\end{aligned} \tag{10}$$

$$\begin{aligned}
& r_1\frac{\partial^4\Psi_1}{\partial z^4} + r_2\frac{\partial^3\Psi_1}{\partial z^3} + r_3\frac{\partial^2\Psi_1}{\partial z^2} + r_4\frac{\partial\Psi_1}{\partial z} + r_5\frac{\partial^4\Psi_1}{\partial z^2\,\partial\theta_1^2} + r_6\frac{\partial^3\Psi_1}{\partial z\,\partial\theta_1^2} + r_7\frac{\partial^2\Psi_1}{\partial\theta_1^2} + r_8\frac{\partial^4\Psi_1}{\partial\theta_1^4} \\
& = e^{-2z}\begin{bmatrix}
b_{14}\dfrac{\partial^4 w}{\partial\theta_1^4} - (2b_{32} - b_{13} - b_{24})\dfrac{\partial^4 w}{\partial z^2\,\partial\theta_1^2} - (3b_{24} - 4b_{32} + b_{13})\dfrac{\partial^3 w}{\partial z\,\partial\theta_1^2} \\
-2(b_{32} - b_{24} - b_{14})\dfrac{\partial^2 w}{\partial\theta_1^2} + b_{23}\dfrac{\partial^4 w}{\partial z^4} - (b_{13} - b_{24} + 4b_{23})\dfrac{\partial^3 w}{\partial z^3} \\
-(-3b_{13} + 3b_{24} - 5b_{23} + b_{14} + S_1e^z\cot\gamma)\dfrac{\partial^2 w}{\partial z^2} \\
-(2b_{13} + 2b_{23} - 2b_{14} - 2b_{24} - S_1e^z\cot\gamma)\dfrac{\partial w}{\partial z}
\end{bmatrix}
\end{aligned} \tag{11}$$

in which expressions c_{ij}, b_{ij}, r_l ($i, j = 1$–4, $l = 1$–8) are defined as follows:

$$\begin{aligned}
& c_{11} = a_{11}^1 b_{11} + a_{12}^1 b_{21}, \quad c_{12} = a_{11}^1 b_{12} + a_{12}^1 b_{22}, \quad c_{13} = a_{11}^1 b_{13} + a_{12}^1 b_{23} + a_{11}^2 \\
& c_{14} = a_{11}^1 b_{14} + a_{12}^1 b_{24} + a_{12}^2, \quad c_{21} = a_{21}^1 b_{11} + a_{22}^1 b_{21}, \quad c_{22} = a_{21}^1 b_{12} + a_{22}^1 b_{22} \\
& c_{23} = a_{21}^1 b_{13} + a_{22}^1 b_{14} + a_{21}^2, \quad c_{24} = a_{21}^1 b_{14} + a_{22}^1 b_{13} + a_{22}^2, \quad c_{31} = a_{33}^1 b_{31} \\
& c_{32} = a_{33}^1 b_{32} + a_{33}^2, \quad r_1 = b_{22}, \quad r_2 = b_{21} - b_{12} - 4b_{22}, \quad r_3 = -3b_{21} - b_{11} + 5b_{22} + 3b_{12} \\
& r_4 = 2(b_{11} + b_{21} - b_{12} - b_{22}), \quad r_5 = 2b_{31} + b_{21} + b_{12}, \quad r_6 = -4b_{31} - 3b_{12} - b_{21} \\
& r_7 = 2(b_{31} + b_{21} + b_{11}), \quad r_8 = b_{11}, \quad b_{11} = a_{22}^0 L_0^{-1}, \quad b_{12} = -a_{12}^0 L_0^{-1} \\
& b_{13} = (a_{12}^0 a_{21}^1 - a_{11}^1 a_{22}^0)L_0^{-1}, \quad b_{14} = (a_{12}^0 a_{22}^1 - a_{12}^1 a_{22}^0)L_0^{-1}, \quad b_{21} = -a_{21}^0 L_0^{-1} \\
& b_{22} = -a_{22}^0 L_0^{-1}, \quad b_{23} = (a_{21}^0 a_{11}^1 - a_{21}^1 a_{11}^0)L_0^{-1}, \quad b_{24} = (a_{21}^0 a_{12}^1 - a_{22}^1 a_{11}^0)L_0^{-1} \\
& b_{31} = 1/a_{66}^0, \quad b_{32} = -a_{66}^1/a_{66}^0, \quad L_0 = a_{11}^0 a_{11}^0 - a_{12}^0 a_{12}^0
\end{aligned} \tag{12}$$

in which expressions $a_{11}^k, a_{12}^k, a_{66}^k$ $(k = 0, 1, 2)$ are defined as follows:

$$a_{11}^k = \frac{E_{0S}h^{k+1}}{1-\nu_{S\theta}\nu_{\theta S}}\int_{-1/2}^{1/2}\bar{\zeta}^k\left[1+\mu\varphi_1(\bar{\zeta})\right]\mathrm{d}\bar{\zeta},\quad a_{22}^k = \frac{E_{0\theta}h^{k+1}}{1-\nu_{S\theta}\nu_{\theta S}}\int_{-1/2}^{1/2}\bar{\zeta}^k\left[1+\mu\varphi_1(\bar{\zeta})\right]\mathrm{d}\bar{\zeta}$$

$$a_{12}^k = \nu_{\theta S}a_{11}^k = a_{21}^k = \nu_{S\theta}a_{22}^k,\quad a_{66}^k = 2G_0h^{k+1}\int_{-1/2}^{1/2}\bar{\zeta}^k\left[1+\mu\varphi_1(\bar{\zeta})\right]\mathrm{d}\bar{\zeta}$$

$$\rho_t = \rho_0\int_{-1/2}^{1/2}\left[1+\mu\varphi_2(\bar{\zeta})\right]\mathrm{d}\bar{\zeta},\quad k=0,1,2 \tag{13}$$

3. The solution of basic equations

Consider a non-homogenous orthotropic truncated conical shell with simply supported edge conditions. The solution of Eq. (11) is sought in the following form [35]:

$$w = f(t)\mathrm{e}^z\sin(\beta_1 z)\sin(\beta_2\theta_1) \tag{14}$$

where $f(t)$ is the time-dependent amplitude and the following definitions apply:

$$\beta_1 = \frac{m\pi}{z_0},\quad \beta_2 = \frac{n}{\sin\gamma},\quad z_0 = \ln\frac{S_2}{S_1} \tag{15}$$

Function (14) satisfies the periodical conditions of the normal displacements and all orders of the derivatives for normal displacements and the following geometrical boundary conditions [35]:

$$w(S,\theta) = 0,\quad \frac{\partial^2 w(S,\theta)}{\partial S^2} = 0 \quad \text{at } S = S_1 \quad \text{and} \quad S = S_2 \tag{16}$$

Substituting expression (14) into Eq. (11) and applying the superposition method to the resulting equation, the particular solution is obtained as follows:

$$\Psi_1 = f(t)\left[K_1\sin(\beta_1 z) + K_2\cos(\beta_1 z) + K_3\mathrm{e}^{-z}\sin(\beta_1 z) + K_4\mathrm{e}^{-z}\cos(\beta_1 z)\right]\sin(\beta_2\theta_1) \tag{17}$$

where the following definitions apply:

$$K_1 = \frac{\beta_1(\beta_1 x_0 + y_0)S_1\cot\gamma}{x_0^2+y_0^2},\quad K_2 = \frac{\beta_1(\beta_1 y_0 - x_0)S_1\cot\gamma}{x_0^2+y_0^2}$$
$$K_3 = \frac{z_1x_1 + y_1z_2}{x_1^2+y_1^2},\quad K_4 = \frac{x_1z_2 - z_1y_1}{x_1^2+y_1^2} \tag{18}$$

$$x_0 = -r_3\beta_1^2 + r_5\beta_1^2\beta_2^2 - r_7\beta_2^2 + r_1\beta_1^4 + r_8\beta_2^4$$
$$y_0 = r_2\beta_1^3 - r_4\beta_1 + r_6\beta_1\beta_2^2$$
$$x_1 = r_5\beta_1^2\beta_2^2 + (3r_2 - 6r_1 - r_3)\beta_1^2 + (r_6 - r_7 - r_5)\beta_2^2 + r_1\beta_1^4 + r_8\beta_2^4 + r_1 - r_2 + r_3 - r_4$$
$$y_1 = (r_6 - 2r_5)\beta_1\beta_2^2 + (2r_3 - r_4 + 4r_1 - 3r_2)\beta_1 + (r_2 - 4r_1)\beta_1^3 \tag{19}$$

$$z_1 = b_{14}\left[(\beta_2^2-1)^2 + \beta_1^2\right] + \beta_1^2\left[(b_{13} - 2b_{32} + b_{24})\beta_2^2 + b_{23}(\beta_1^2+1)\right]$$
$$z_2 = (b_{24} - b_{13})\beta_1(\beta_2^2 - \beta_1^2 - 1) \tag{20}$$

Substituting Eqs. (14) and (17) into Eq. (10) and applying the Galerkin method, we get

$$\int_0^{z_0}\int_0^{2\pi\sin\gamma}\Theta\mathrm{e}^z\sin(\beta_1 z)\sin(\beta_2\theta_1)\,\mathrm{d}z\,\mathrm{d}\theta_1 = 0 \tag{21}$$

where the following definition apply:

$$\Theta = f(t)\begin{bmatrix} U_1 e^{2z}\sin(\beta_1 z)\sin(\beta_2\theta_1) + U_2 e^{z}\sin(\beta_1 z)\sin(\beta_2\theta_1) \\ +U_3 e^{2z}\cos(\beta_1 z)\sin(\beta_2\theta_1) + U_4 e^{z}\cos(\beta_1 z)\sin(\beta_2\theta_1) \\ -(K_2\beta_1^2 + K_1\beta_1)S_1 e^{3z}\cos(\beta_1 z)\sin(\beta_2\theta_1)\cot\gamma \\ +(K_2\beta_1 - K_1\beta_1^2)S_1 e^{3z}\sin(\beta_1 z)\sin(\beta_2\theta_1)\cot\gamma \\ -S_1^3 e^{4z}P(1-0.5\beta_1^2-\beta_2^2)\sin(\beta_1 z)\sin(\beta_2\theta_1)\tan\gamma \\ -1.5\beta_1 S_1^3 e^{4z}P\cos(\beta_1 z)\sin(\beta_2\theta_1)\tan\gamma - S_1^4 e^{5z}\sin(\beta_1 z)\sin(\beta_2\theta_1)\dfrac{d^2}{dt^2} \end{bmatrix} \tag{22}$$

in which

$$\begin{aligned} U_1 = {} & K_1 c_{12}\beta_1^4 + K_2\beta_1^3(c_{11}-c_{22}+4c_{12}) - K_1\beta_1^2(5c_{12}+3c_{11}-3c_{22}-c_{21}) \\ & - 2K_2\beta_1(c_{11}-c_{22}+c_{12}-c_{21}) + K_1\beta_2^4 c_{21} + K_1\beta_1^2\beta_2^2(c_{11}-2c_{31}+c_{22}) \\ & + K_2\beta_1\beta_2^2(c_{11}-4c_{31}+3c_{22}) - 2K_1\beta_2^2(c_{22}-c_{31}+c_{21}) - (K_3\beta_1^2+K_4\beta_1)S_1\cot\gamma \end{aligned} \tag{23.1}$$

$$\begin{aligned} U_2 = {} & (K_3\beta_1^4 + K_3\beta_1^2)c_{12} + (K_3\beta_1^2 + K_3 + K_3\beta_2^4 - 2K_3\beta_2^2)c_{21} \\ & + (K_4\beta_1^3 + K_4\beta_1 - K_4\beta_1\beta_2^2)(c_{11}-c_{22}) + K_3\beta_1^2\beta_2^2(c_{11}-2c_{31}+c_{22}) \\ & - c_{13}\beta_1^4 - (c_{13}+c_{24})\beta_1^2 - c_{24} - c_{24}\beta_2^4 - \beta_1^2\beta_2^2(c_{14}+c_{23}+2c_{32}) + 2c_{24}\beta_2^2 \end{aligned} \tag{23.2}$$

$$\begin{aligned} U_3 = {} & K_2 c_{12}\beta_1^4 - K_1\beta_1^3(c_{11}-c_{22}+4c_{12}) - K_2\beta_1^2(5c_{12}+3c_{11}-3c_{22}-c_{21}) \\ & + 2K_1\beta_1(c_{11}-c_{22}+c_{12}-c_{21}) + K_2\beta_1^2\beta_2^2(c_{11}-2c_{31}+c_{22}) + K_2\beta_2^4 c_{21} \\ & - K_1\beta_1\beta_2^2(3c_{22}-4c_{31}+c_{11}) - 2K_2\beta_2^2(c_{22}-c_{31}+c_{21}) + (K_3\beta_1 - K_4\beta_1^2)S_1\cot\gamma \end{aligned} \tag{23.3}$$

$$\begin{aligned} U_4 = {} & K_4(\beta_1^4+\beta_1^2)c_{12} + K_4(\beta_1^2+1+\beta_2^4-2\beta_2^2)c_{21} + K_3(\beta_1\beta_2^2-\beta_1^3-\beta_1)(c_{11}-c_{22}) \\ & + K_4\beta_1^2\beta_2^2(c_{11}-2c_{31}+c_{22}) + (c_{14}-c_{23})\left[\beta_1+\beta_1^3-\beta_1\beta_2^2\right] \end{aligned} \tag{23.4}$$

After substituting Eq. (22) into Eq. (21), and integrating it, the equation yields

$$\frac{d^2 f(t)}{dt^2} + \frac{\Gamma_{10}}{\Gamma_9\rho_t h}\left[\frac{\Gamma_1 U_1 + \Gamma_2 U_2 + \Gamma_3 U_3 + \Gamma_4 U_4 + \Gamma_8}{\Gamma_{10}} - P\right] f(t) = 0 \tag{24}$$

where the following definitions apply:

$$\begin{aligned} & \Gamma_1 = \frac{2\beta_1^2}{12\beta_1^2+27}\left[1-\left(\frac{S_2}{S_1}\right)^3\right], \quad \Gamma_2 = \frac{\beta_1^2}{4(\beta_1^2+1)}\left[1-\left(\frac{S_2}{S_1}\right)^2\right], \quad \Gamma_3 = \frac{\beta_1}{4\beta_1^2+9}\left[\left(\frac{S_2}{S_1}\right)^3-1\right] \\ & \Gamma_4 = \frac{\beta_1}{4(\beta_1^2+1)}\left[\left(\frac{S_2}{S_1}\right)^2-1\right], \quad \Gamma_5 = \frac{\beta_1^2}{8(\beta_1^2+4)}\left[1-\left(\frac{S_2}{S_1}\right)^4\right], \quad \Gamma_6 = \frac{\beta_1}{4(\beta_1^2+4)}\left[\left(\frac{S_2}{S_1}\right)^4-1\right] \\ & \Gamma_7 = \frac{\beta_1^2}{12(\beta_1^2+9)}\left[\left(\frac{S_2}{S_1}\right)^6-1\right], \quad \Gamma_{10} = \frac{S_1^3\beta_1^2(2\beta_1^2+4\beta_2^2+11)\tan\gamma}{10(25+4\beta_1^2)}\left[\left(\frac{S_2}{S_1}\right)^5-1\right] \\ & \Gamma_8 = \left[(2K_1 - 3K_2\beta_1 - K_1\beta_1^2)\Gamma_5 - (K_2\beta_1^2 - 3K_1\beta_1 - 2K_2)\Gamma_6\right]S_1\cot\gamma, \quad \Gamma_9 = S_1^4\Gamma_7 \end{aligned} \tag{25}$$

When $P = 0$, the following expression is obtained from Eq. (24) for the frequency of free vibration:

$$\omega = \left(\frac{\Gamma_1 U_1 + \Gamma_2 U_2 + \Gamma_3 U_3 + \Gamma_4 U_4 + \Gamma_8}{\Gamma_9\rho_t h}\right)^{0.5} \tag{26}$$

The minimum value of the frequency parameter is obtained by minimizing Eq. (26) with respect to m and n.

The cyclic natural frequency (Hz) of the shell is defined as

$$\overline{\omega} = \omega/(2\pi) \tag{27}$$

The dimensionless frequency parameter ω_1 is defined as

$$\omega_1 = \omega R_2 \left[(1 - \nu_{\theta S}\nu_{S\theta})\rho_0/E_{0\theta}\right]^{0.5} \tag{28}$$

At the static case, for the hydrostatic buckling pressure, from Eq. (24) the following equation is obtained:

$$P_{\mathrm{cr}} = \frac{U_1\Gamma_1 + U_2\Gamma_2 + U_3\Gamma_3 + U_4\Gamma_4 + \Gamma_8}{\Gamma_{10}} \tag{29}$$

The dimensionless hydrostatic buckling pressure $P_{1\mathrm{cr}}$ is defined as

$$P_{1\mathrm{cr}} = \frac{P_{\mathrm{cr}}}{E_{0\theta}} \tag{30}$$

The minimum value of the dimensionless hydrostatic buckling pressure is obtained by minimizing Eq. (30) with respect to m and n, the number of longitudinal and circumferential buckling waves, respectively.

4. Numerical computations and results

4.1. Comparative problems

In order to verify the accuracy of the present buckling analysis, eight comparisons are made with the results existing in the open literature.

As shown in Table 1, the first example is about the buckling of a homogeneous isotropic conical shell under hydrostatic pressure by simply taking $\mu = 0$, $E_{0S} = E_{0\theta} = E_0$ and $\nu_{\theta S} = \nu_{S\theta} = \nu_0$ in the present formulations in order to convert the non-homogeneous orthotropic conical shell formulation into a homogeneous isotropic conical shell. A close examination of Table 1 shows that there are good agreements between the present formulations and the results of Niordson [25], Mushtari and Sachenkov [26], Seide [28] and Singer [29]. The numbers in brackets indicate the buckling mode (m, n).

The second example is about the buckling of a homogeneous orthotropic conical shell under hydrostatic pressure. The present non-homogeneous orthotropic conical shell formulation is converted into a homogeneous orthotropic conical shell by simply letting $\mu = 0$. For a homogeneous orthotropic conical shells of fiber glass reinforced epoxy having $\gamma = 30^\circ$ and 75°, $S_1 = 57.59$ Inc., $S_2 = 86.385$ Inc., $h = 0.1$ Inc. Table 2 shows the comparison of present results with those reported by Singer and Fershst-Scher [39] and a good agreement is achieved. In addition, the buckling loads for simply supported, homogeneous isotropic cylindrical shells under a hydrostatic pressure are calculated and compared in Table 3 with theoretical results of Hutchinson and Amazigo [68], finite element results obtained by Kasagi and Sridharan [69], and boundary layer theory solution of Shen and Noda [70]. As shown in Table 3 (identical in Tables 5 and 6), is concerned

Table 1
Comparison of various results for the hydrostatic buckling pressure of conical shells ($S_1 = 57.59$ Inc.; $h = 0.1$ Inc.; $\nu_0 = 0.3$)

γ (deg)	S_2/S_1	$P_{1\mathrm{cr}} \times 10^6$ and (m, n)				
		Niordson [25]	Mushtari and Sachenkov [26]	Seide [28]	Singer [29]	Present study
10	1.5	2.29	2.35	2.35	2.35	2.284 (1,6)
10	2.5	0.484	0.539	0.534	0.545	0.471 (1,4)
30	1.5	0.396	0.406	0.405	0.405	0.393 (1,12)
30	4.0	0.0225	0.0332	0.0284	0.0272	0.0219 (1,8)
30	5.0	0.0128	0.0215	0.0169	0.0155	0.0121(1,8)
30	10.0	0.00228	0.00617	0.00368	0.00278	0.00191(1,8)

Table 2
Comparisons of dimensionless buckling loads $P_{1\mathrm{cr}}$ for homogeneous orthotropic conical shells under a hydrostatic pressure ($S_1 = 57.59$ Inc., $S_2 = 86.385$ Inc., $h = 0.1$ Inc.)

$P_{1\mathrm{cr}} \times 10^6$ and (m, n)				
γ (deg)	$E_{0S}/E_{0\theta}$	$v_{\theta S}$	Singer and Fershst-Scher [39]	Present study
30	2.591	0.090	0.2041	0.1877 (1,14)
30	0.386	0.234	0.7732	0.732 (1,11)
75	2.591	0.090	0.0142	0.0129 (1,15)
75	0.386	0.234	0.0487	0.0450 (1,14)

Table 3
Comparisons of buckling loads P_{cr} (psi) for homogeneous isotropic cylindrical shells under a hydrostatic pressure ($R/h = 200$)

$\overline{Z}_B$	Shen and Noda [70]	Hutchinson and Amazigo [68] (theoretical study)	Kasagi and Sridharan [69] (FEM study)	Present study
10	87.077 (1,18)	89.07 (1,18)	88.65 (1,18)	88.952 (1,18)
50	35.167 (1,13)	35.25 (1,13)	35.09 (1,13)	35.211 (1,13)
100	24.305 (1,11)	24.35 (1,11)	24.26 (1,11)	24.320 (1,11)
500	10.436 (1,8)	10.45 (1,8)	10.42 (1,8)	10.440 (1,8)
1000	7.398 (1,7)	7.412 (1,7)	7.388 (1,7)	7.400 (1,7)
5000	3.416 (1,5)	3.423 (1,5)	3.412 (1,5)	3.416 (1,5)
10,000	2.315 (1,4)	2.319 (1,4)	2.312 (1,4)	2.315 (1,4)

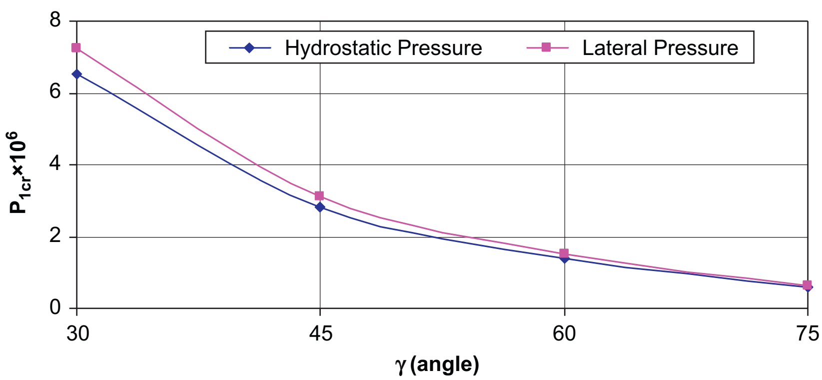

Fig. 2. Comparison of various results for dimensionless hydrostatic and lateral buckling pressures of homogeneous orthotropic truncated conical shells.

about buckling of a homogeneous isotropic cylindrical shell by taking $\gamma = \pi/180{,}000 \approx 0^\circ$, $\mu = 0$, $E_{0S} = E_{0\theta} = E_0$ and $v_{\theta S} = v_{S\theta} = v_0$ in the present formulations; and then the non-homogeneous orthotropic conical shell becomes a homogeneous isotropic cylindrical shell. Furthermore, $\overline{Z}_B = [L_1^2/(Rh)](1 - v_0^2)^{1/2}$ is the Batdorf cylindrical shell parameter [70]. Here, R and L_1 are the radius and length of cylindrical shell, respectively. The material properties are, as stated by Hutchinson and Amazigo [68], $E_0 = 10 \times 10^6$ psi, $v_0 = 0.33$. The numbers in brackets indicate the buckling mode (m, n). It is clear that, for most cases the present results agree well with the existing results. The critical values of the hydrostatic and the lateral pressures and the corresponding wavenumbers for different semi-vertex angle for the orthotropic truncated conical shells are given in Fig. 2. Computational data for the material properties are $E_{0S} = 1.724 \times 10^{11}\,\mathrm{N\,m^{-2}}$, $E_{0\theta} = 7.79 \times 10^9\,\mathrm{N\,m^{-2}}$, $v_{\theta S} = 0.35$, $\rho_0 = 1530\,\mathrm{kg\,m^{-3}}$ shell geometry properties are $R_2/h = 500$ and $L = 0.25 R_2 \sin\gamma$. As shown in Fig. 2, the critical value of hydrostatic pressure is quite different from the

Table 4
Comparisons of frequency parameter ω_1 of a homogeneous isotropic truncated conical shell with simply supported at both ends ($v_0 = 0.3$, $h/R_2 = 0.01$, $L = 0.25S_2$)

n	Tong [57]			Lam and Li [59] and Li [60]		
	30°	45°	60°	30°	45°	60°
2	0.7910	0.6879	0.5720	0.8420	0.7655	0.6348
3	0.7284	0.6973	0.5998	0.7376	0.7212	0.6238
4	0.6353	0.6664	0.6048	0.6362	0.6739	0.6145
5	0.5531	0.6304	0.6069	0.5528	0.6323	0.6111
6	0.4949	0.6032	0.6147	0.4950	0.6035	0.6171
7	0.4643	0.5918	0.6329	0.4661	0.5921	0.6350
8	0.4644	0.5992	0.6632	0.4660	0.6001	0.6660
9	0.4892	0.6257	0.7060	0.4916	0.6273	0.7101

n	Liew et al. [53]	Irie et al. [48]			Present study		
	30°	30°	45°	60°	30°	45°	60°
2	0.7904	0.7910	0.6879	0.5722	0.7943	0.7169	0.6017
3	0.7274	0.7284	0.6973	0.6001	0.7085	0.6832	0.5959
4	0.6339	0.6352	0.6664	0.6054	0.6199	0.6462	0.5922
5	0.5514	0.5531	0.6304	0.6077	0.5437	0.6130	0.5937
6	0.4930	0.4949	0.6032	0.6159	0.4896	0.5901	0.6036
7	0.4632	0.4643	0.5918	0.6343	0.4623	0.5824	0.6240
8	0.4623	0.4645	0.5992	0.6650	0.4627	0.5924	0.6563
9	0.4870	0.4892	0.6257	0.7084	0.4882	0.6204	0.7009

Table 5
Comparison of $\overline{\omega}$ (Hz) of an isotropic cylindrical shell ($h = 0.000648\,\text{m}$, $R = 0.2423\,\text{m}$, $L = 0.6096\,\text{m}$, $E_0 = 6.895 \times 10^4\,\text{MPa}$, $v_0 = 0.315$, $m = 1$, $\rho_0 = 2714.5\,\text{kg}\,\text{m}^{-3}$)

n	Naem and Sharma [72] with $N_1 = 8$	Sewal and Naumann [71] (experimental study)	Present study
4	287.59	287.0	297.70
5	201.85	203.0 and 211.0	207.07
6	166.59	175.0	170.64
7	166.22	163.0 and 169.0	170.21
8	189.29	188.0	193.37
9	226.88	224.0	231.03
10	274.09	268.0	278.28
11	328.64	326.0	332.85
12	389.49	382.0 and 385.0	393.74
13	456.21	440.0	460.49
14	528.56	–	532.87
15	606.45	590.0	610.78

critical value of lateral pressure. For example, in the case of $\gamma = 45°$, percent difference between the critical values of hydrostatic and the lateral pressures is 9%.

In order to examine the accuracy of the present free vibration analysis, seven comparisons are made with the results available in the literature.

Table 4 shows the dimensionless frequency parameters ω_I for a homogeneous isotropic conical shell with $h/R_2 = 0.01$, $L = 0.25S_2$ and different γ values, with longitudinal wavenumber $m = 1$. The present results are compared with solutions given by Irie et al. [48], Tong [57], Lam and Li [59], Li [60] and Liew et al. [53] and a good agreement is obtained for all the modes. In the study of Liew et al. [53], for the free vibration frequency,

Table 6
Comparison of ϖ (Hz) of an isotropic cylindrical shell ($h = 0.000247\,\text{m}$, $L/R = 2$, $E_0 = 71.02 \times 10^9\,\text{N}\,\text{m}^{-2}$, $\nu_0 = 0.31$, $m = 1$, $\rho_0 = 2796\,\text{kg}\,\text{m}^{-3}$)

n	Pellicano [73] (theoretical study)	Pellicano [73] (FEM study)	Present study
6	553.3	553.3	564.1
7	484.6	484.6	493.7
8	489.6	489.6	498.5
9	546.2	546.2	555.3
10	636.8	636.8	645.9
11	750.7	750.7	759.8
12	882.2	882.2	891.4

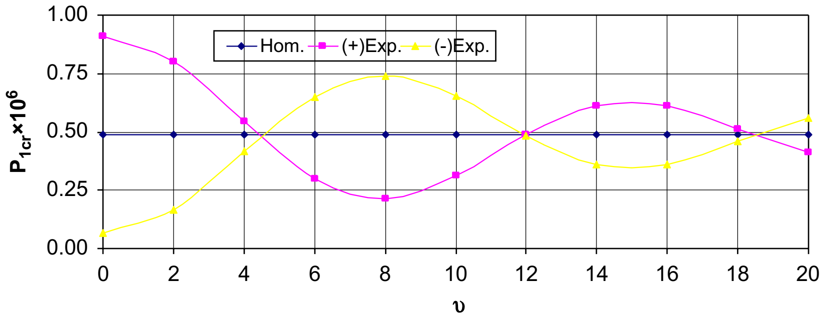

Fig. 3. Variation of P_{1cr} with the non-homogeneity parameter υ for $\varphi_1(\bar{\zeta})$.

the parameters are the same and calculation were carried on only for $\gamma = 30^\circ$, and the tabulated results exists in Table 4.

A comparison of the natural frequencies for a homogeneous isotropic cylindrical shell with those available in the literature obtained by various methods has been presented in Tables 5 and 6, where $\varpi = \omega/(2\pi)$ (Hz) the cyclic natural frequencies of the shell and N_1 are show the number of polynomials. Table 5 presents the comparison of the natural frequencies ϖ(Hz) for a homogeneous isotropic cylindrical shell, with those obtained in Ref. [71] experimentally and those obtained in Ref. [72] analytically. It is observed that there is a good match between the results of the two studies. The highest percentage of the difference between the two being 5.5%, it corresponds to the circumferential wavenumber $n = 4$. Table 6 shows the comparison of the natural frequencies ϖ(Hz) for a homogeneous isotropic cylindrical shell, with those obtained in Ref. [73] by using finite element method and theoretically. It is observed that there is a good match between the results of studies. The lowest and highest percentages of the difference between the two beings are 1% and 1.9%, respectively.

Based on the above comparisons of Tables 1–6, the accuracy of the present study is validated.

4.2. Vibration and buckling analyses

Numerical computations, for homogeneous and non-homogeneous orthotropic truncated conical shells have been carried out using expressions (28) and (30). The results are presented in Figs. 3–10 and Tables 7–10.

Homogeneous and non-homogeneous orthotropic truncated conical shells with different types of geometry are considered and their critical dimensionless hydrostatic pressure and lowest frequency parameter are computed. Values of the critical parameters are obtained for $10 \leqslant Z_B \leqslant 150$, i.e., for short shells; for $150 < Z_B < 1500$, i.e., for medium length shells; for $1500 \leqslant Z_B \leqslant 5000$, i.e., for long shells and for $Z_B > 5000$, i.e., for very long shells. Here, $Z_B = 2L^2(1 - \nu_0^2)^{1/2} \cos\gamma/(R_1 + R_2)h$ is the Batdorf conical shell parameter [32].

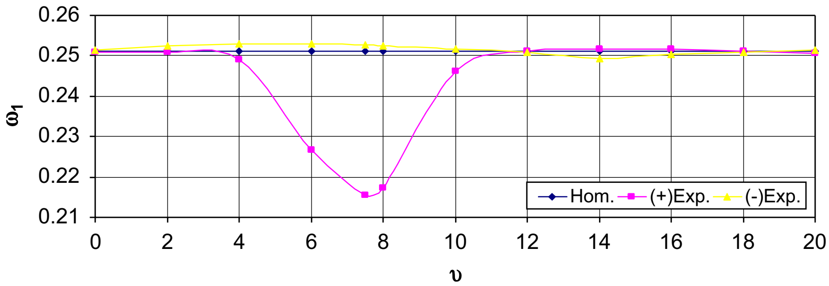

Fig. 4. Variation of frequency parameter ω_1 with the non-homogeneity parameter υ for $\varphi_j(\bar{\zeta})$ $(j = 1, 2)$.

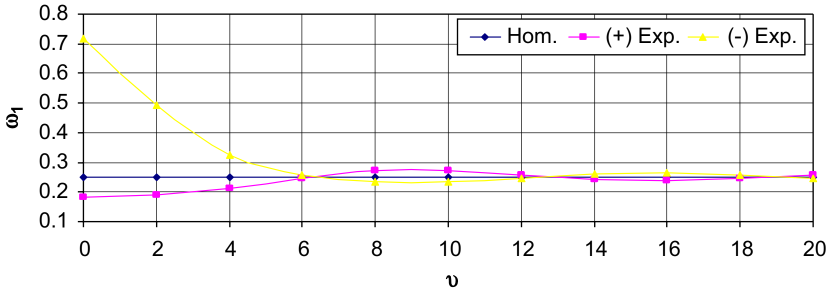

Fig. 5. Variation of frequency parameter ω_1 with the non-homogeneity parameter υ for $\varphi_2(\bar{\zeta})$.

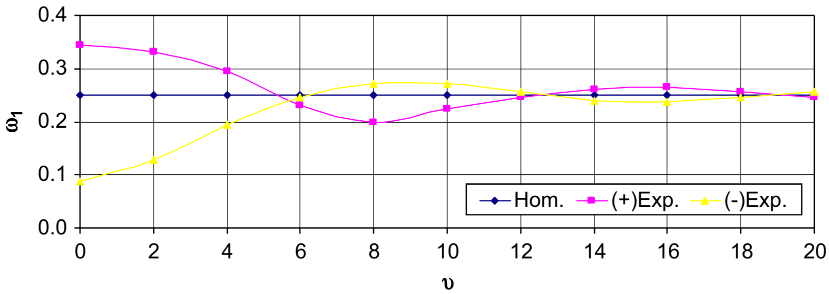

Fig. 6. Variation of frequency parameter ω_1 with the non-homogeneity parameter υ for $\varphi_1(\bar{\zeta})$.

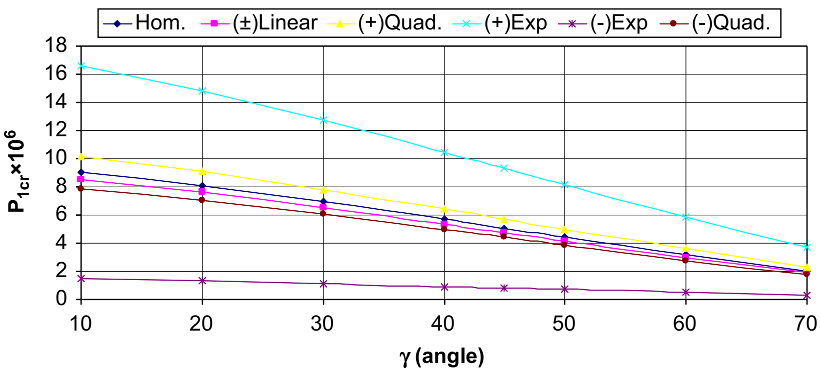

Fig. 7. Variations of the values of P_{1cr} for homogeneous and non-homogeneous orthotropic, short truncated conical shells with respect to semi-vertex angles γ ($R_1/h = 200$, $L/R_1 = 0.5$, $\mu = 0.9$, $j = 1$).

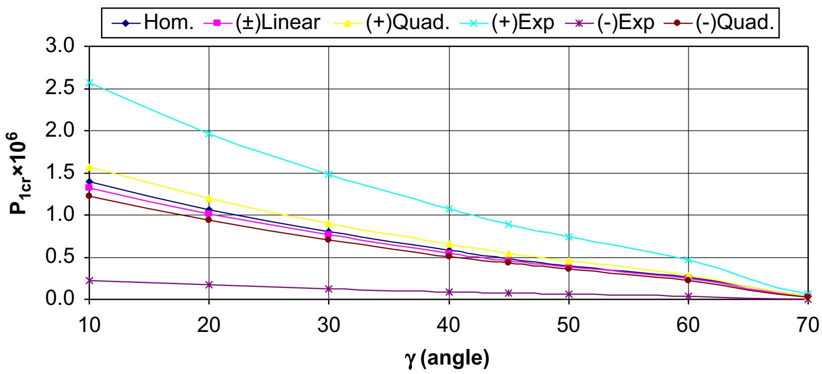

Fig. 8. Variations of the values of P_{1cr} for homogeneous and non-homogeneous orthotropic, medium length, truncated conical shells with respect to semi-vertex angles γ ($R_1/h = 200$, $L/R_1 = 2$, $\mu = 0.9$, $j = 1$).

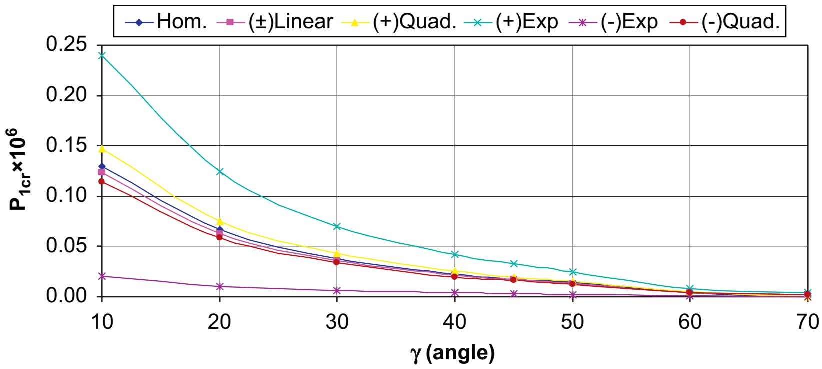

Fig. 9. Variations of the values of P_{1cr} for homogeneous and non-homogeneous orthotropic, long truncated conical shells with respect to semi-vertex angles γ ($R_1/h = 200$, $L/R_1 = 10$, $\mu = 0.9$, $j = 1$).

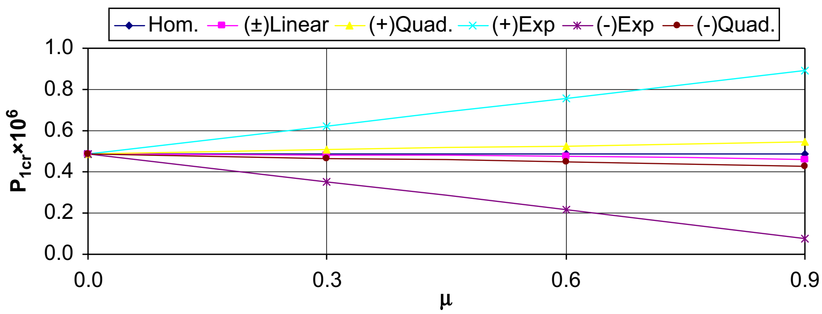

Fig. 10. Variations of the values of P_{1cr} for homogeneous and non-homogeneous orthotropic truncated conical shells versus μ ($L/R_1 = 2$, $\gamma = 45^\circ$, $R_1/h = 200$, $j = 1$).

Composite material properties are given below [66,67]:

$$E_{0S} = 1.724 \times 10^{11}\,\mathrm{N\,m^{-2}}, \quad E_{0\theta} = 7.79 \times 10^{9}\,\mathrm{N\,m^{-2}}, \quad \nu_{\theta S} = 0.35, \quad \rho_0 = 1530\,\mathrm{kg\,m^{-3}}$$

Numerical analyses are realized to define the change intervals of non-homogeneity parameter υ, and to determine the effects on critical parameters when Young's moduli and density, as together and separately, vary as exponential function. Figs. 3–6 illustrate the results of the numerical analyses.

Table 7
Variations of the values of P_{1cr} and n_{cr} for homogeneous and non-homogeneous orthotropic truncated conical shells with ratios L/R_1 ($\gamma = 45^\circ$, $R_1/h = 200$, $\mu = 0.9$, $j = 1$)

$\varphi_j(\bar{\zeta})$	Hom.	$\pm\bar{\zeta}$	$\bar{\zeta}^2$	$\bar{\zeta}^3$
L/R_1	$P_{1cr} \times 10^6$ and (n_{cr})			
0.25	18.558 (28)	17.339 (28)	21.0323 (28)	18.530 (28)
0.50	5.065 (20)	4.758 (20)	5.711 (19)	5.058 (20)
0.75	2.537 (17)	2.395 (17)	2.854 (17)	2.534 (17)
1.0	1.574 (16)	1.488 (16)	1.770 (16)	1.572 (16)
2.0	0.486 (14)	0.460 (14)	0.546 (14)	0.485 (14)
3.0	0.231 (13)	0.219 (13)	0.259 (13)	0.231 (13)
4.0	0.133 (13)	0.126 (13)	0.149 (12)	0.133 (13)
5.0	0.084 (12)	0.080 (12)	0.094 (12)	0.084 (12)
6.0	0.057 (12)	0.054 (12)	0.064 (12)	0.057 (12)
7.0	0.040 (12)	0.038 (12)	0.045 (12)	0.040 (12)
8.0	0.030 (12)	0.028 (12)	0.033 (12)	0.030 (12)
9.0	0.023 (12)	0.022 (12)	0.025 (12)	0.023 (12)
10	0.018 (12)	0.017 (12)	0.020 (12)	0.018 (12)
20	0.002 (9)	0.002 (9)	0.002 (9)	0.002 (9)
$\varphi_j(\bar{\zeta})$	$e^{-0.1\lvert\bar{\zeta}\rvert}\cos(0.7\bar{\zeta})$	$-e^{-0.1\lvert\bar{\zeta}\rvert}\cos(0.7\bar{\zeta})$	$-\bar{\zeta}^2$	$-\bar{\zeta}^3$
L/R_1	$P_{1cr} \times 10^6$ and (n_{cr})			
0.25	34.074 (28)	3.042 (28)	16.083 (28)	18.530 (28)
0.50	9.309 (20)	0.818 (19)	4.412 (20)	5.058 (20)
0.75	4.667 (17)	0.407 (17)	2.220 (17)	2.534 (17)
1.0	2.897 (16)	0.252 (16)	1.379 (16)	1.572 (16)
2.0	0.894 (14)	0.077 (13)	0.426 (14)	0.485 (14)
3.0	0.426 (13)	0.037 (13)	0.203 (13)	0.231 (13)
4.0	0.245 (13)	0.021 (12)	0.117 (13)	0.133 (13)
5.0	0.154 (12)	0.013 (12)	0.074 (12)	0.084 (12)
6.0	0.105 (12)	0.009 (12)	0.050 (12)	0.057 (12)
7.0	0.074 (12)	0.006 (12)	0.036 (12)	0.040 (12)
8.0	0.055 (12)	0.005 (12)	0.026 (12)	0.030 (12)
9.0	0.042 (12)	0.004 (11)	0.020 (12)	0.023 (12)
10	0.033 (12)	0.003 (11)	0.016 (12)	0.018 (12)
20	0.003 (9)	0.000270 (9)	0.001 (9)	0.002 (9)

In Fig. 3 are seen three curves pertaining to the critical dimensionless hydrostatic pressure for three different forms of Young's moduli variation functions. In curve 1, the case of $\varphi_1(\bar{\zeta}) = e^{-0.1\lvert\bar{\zeta}\rvert}\cos(\upsilon\bar{\zeta})$ and $\mu = 0.90$ is considered. For $0 \leqslant \upsilon \leqslant 4.5$, the critical dimensionless hydrostatic pressure is higher than that for the homogeneous case and takes its maximum value for $\upsilon = 0$ as (0.908×10^{-6}). For $4.5 < \upsilon \leqslant 12$, it is lower than the value for the homogeneous case and takes its minimum value for $\vartheta = 8$ as (0.212×10^{-6}). Curve 2 corresponds to the homogeneous case and the pertinent value of the critical dimensionless hydrostatic pressure is 0.486×10^{-6}. In curve 3, the case of $\varphi_1(\bar{\zeta}) = -e^{-0.1\lvert\bar{\zeta}\rvert}\cos(\upsilon\bar{\zeta})$ and $\mu = 0.90$ is considered. For $0 \leqslant \upsilon \leqslant 4.5$, the critical dimensionless hydrostatic pressure is lower than that for the homogeneous case and takes its minimum value for $\upsilon = 0$ as (0.064×10^{-6}), whereas, for $4.5 < \upsilon \leqslant 12$, it takes higher values and is a maximum for $\upsilon = 8$, taking the value (0.737×10^{-6}). For $4 < \upsilon < 5$ and $\upsilon > 11$, the effect of the variation of Young's moduli on the critical dimensionless hydrostatic pressure is very little (Fig. 3).

In Fig. 4 are shown three curves pertaining to the lowest frequency parameter for three different forms of Young's moduli and density variation functions. In curve 1, the case of $\varphi_j(\bar{\zeta}) = e^{-0.1\lvert\bar{\zeta}\rvert}\cos(\upsilon\bar{\zeta})$ $(j = 1, 2)$ and $\mu = 0.90$ is considered. For $0 \leqslant \upsilon \leqslant 11.6$, the frequency parameter is lower than that for the homogeneous case

Table 8
Variations of the lowest values of ω_1 and n_{1cr} for homogeneous and non-homogeneous orthotropic truncated conical shells with ratios L/R_1 ($\gamma = 45^\circ$, $R_1/h = 200$, $\mu = 0.9$, $j = 1{,}2$)

$\varphi_j(\bar{\zeta})$	Hom. (0)	$\pm\bar{\zeta}$	$\bar{\zeta}^2$	$-\bar{\zeta}^2$	(+exp)	(−exp)
L/R_1	ω_1 and (n_{1cr}) ($j = 1{,}2$)					
0.25	1.467 (1)	1.429 (1)	1.497 (1)	1.431 (1)	1.459 (1)	1.560 (1)
0.50	0.749 (11)	0.737 (11)	0.759 (10)	0.737 (11)	0.747 (11)	0.779 (10)
0.75	0.541 (12)	0.532 (12)	0.548 (12)	0.533 (12)	0.539 (12)	0.563 (11)
1.0	0.435 (12)	0.428 (12)	0.441 (12)	0.428 (12)	0.433 (12)	0.453 (11)
2.0	0.251 (2)	0.251 (2)	0.251 (2)	0.251 (2)	0.251 (2)	0.252 (2)
3.0	0.203 (11)	0.200 (11)	0.206 (11)	0.200 (11)	0.203 (11)	0.211 (10)
4.0	0.168 (10)	0.165 (10)	0.170 (10)	0.165 (10)	0.167 (10)	0.174 (10)
5.0	0.139 (5)	0.139 (5)	0.140 (5)	0.139 (5)	0.139 (5)	0.140 (5)
6.0	0.071 (5)	0.070 (5)	0.071 (5)	0.070 (5)	0.071 (5)	0.071 (5)
7.0	0.113 (10)	0.111 (10)	0.114 (10)	0.111 (10)	0.112 (10)	0.117 (10)
8.0	0.100 (6)	0.100 (6)	0.101 (6)	0.100 (6)	0.100 (6)	0.101 (6)
9.0	0.077 (6)	0.077 (6)	0.077 (6)	0.077 (6)	0.077 (6)	0.078 (6)
10	0.051 (6)	0.051 (6)	0.052 (6)	0.051 (6)	0.051 (6)	0.052 (6)
20	0.031 (9)	0.030 (9)	0.032 (9)	0.030 (9)	0.031 (9)	0.033 (9)
L/R_1	ω_1 and (n_{1cr}) ($j = 1$)					
0.25	1.467 (1)	1.429 (1)	1.552 (1)	1.376 (1)	1.990 (1)	0.583 (1)
0.50	0.749 (11)	0.737 (11)	0.787 (10)	0.709 (11)	1.019 (11)	0.291 (10)
0.75	0.541 (12)	0.532 (12)	0.569 (12)	0.512 (12)	0.736 (12)	0.211 (11)
1.0	0.435 (12)	0.428 (12)	0.457 (12)	0.412 (12)	0.591 (12)	0.169 (11)
2.0	0.251 (2)	0.251 (2)	0.260 (2)	0.241 (2)	0.342 (2)	0.094 (2)
3.0	0.203 (11)	0.200 (11)	0.214 (11)	0.192 (11)	0.276 (11)	0.079 (10)
4.0	0.168 (10)	0.165 (10)	0.176 (10)	0.159 (10)	0.228 (10)	0.065 (10)
5.0	0.139 (5)	0.139 (5)	0.145 (5)	0.134 (5)	0.190 (5)	0.052 (5)
6.0	0.071 (5)	0.070 (5)	0.073 (5)	0.068 (5)	0.096 (5)	0.027 (5)
7.0	0.113 (10)	0.111 (10)	0.118 (10)	0.107 (10)	0.153 (10)	0.044 (10)
8.0	0.100 (6)	0.100 (6)	0.104 (6)	0.096 (6)	0.137 (6)	0.038 (6)
9.0	0.077 (6)	0.077 (6)	0.080 (6)	0.074 (6)	0.105 (6)	0.029 (6)
10	0.051 (6)	0.051 (6)	0.054 (6)	0.049 (6)	0.070 (6)	0.020 (6)
20	0.031 (9)	0.030 (9)	0.033 (9)	0.029 (9)	0.042 (9)	0.013 (9)
L/R_1	ω_1 and (n_{1cr}) ($j = 2$)					
0.25	1.467 (1)	1.467 (1)	1.414 (1)	1.525 (1)	1.075 (1)	3.923 (1)
0.50	0.749 (11)	0.749 (11)	0.723 (11)	0.779 (11)	0.549 (11)	2.005 (11)
0.75	0.541 (12)	0.541 (12)	0.522 (12)	0.563 (12)	0.397 (12)	1.448 (12)
1.0	0.435 (12)	0.435 (12)	0.419 (12)	0.452 (12)	0.319 (12)	1.163 (12)
2.0	0.251 (2)	0.251 (2)	0.242 (2)	0.261 (2)	0.184 (2)	0.672 (2)
3.0	0.203 (11)	0.203 (11)	0.196 (11)	0.211 (11)	0.149 (11)	0.544 (11)
4.0	0.168 (10)	0.168 (10)	0.162 (10)	0.174 (10)	0.123 (10)	0.449 (10)
5.0	0.139 (5)	0.139 (5)	0.135 (5)	0.145 (5)	0.102 (5)	0.373 (5)
6.0	0.071 (5)	0.071 (5)	0.068 (5)	0.073 (5)	0.052 (5)	0.189 (5)
7.0	0.113 (10)	0.113 (10)	0.109 (10)	0.117 (10)	0.083 (10)	0.302 (10)
8.0	0.100 (6)	0.100 (6)	0.097 (6)	0.104 (6)	0.074 (6)	0.269 (6)
9.0	0.077 (6)	0.077 (6)	0.074 (6)	0.080 (6)	0.056 (6)	0.206 (6)
10	0.051 (6)	0.051 (6)	0.049 (6)	0.053 (6)	0.038 (6)	0.137 (6)
20	0.031 (9)	0.031 (9)	0.030 (9)	0.032 (9)	0.023 (9)	0.083 (9)

and takes its minimum value for $\upsilon = 7.5$ as (0.215). For $11.6 < \upsilon \leqslant 18$, it is higher than the value for the homogeneous case and takes its maximum value for $\upsilon = 14$ as (0.252). Curve 2 corresponds to the homogeneous case and the pertinent value of the lowest frequency parameter is 0.251. In curve 3, the case of

Table 9
Variations of the values of P_{1cr}, ω_1 and corresponding wavenumbers for homogeneous and non-homogeneous orthotropic truncated conical shells with respect to R_1/h ($\gamma = 45^\circ$, $L/R_1 = 3$, $\mu = 0.9$)

$\varphi_j(\bar{\zeta})$	Hom.	$\pm\bar{\zeta}$	$\bar{\zeta}^2$	$-\bar{\zeta}^2$	$\pm\bar{\zeta}^3$
R_1/h	$P_{1cr} \times 10^6$ and (n_{cr}) ($j = 1$)				
50	7.828 (8)	7.414 (9)	8.752 (8)	6.874 (9)	7.820 (8)
100	1.344 (11)	1.270 (11)	1.508 (10)	1.177 (11)	1.342 (11)
150	0.478 (12)	0.453 (12)	0.537 (12)	0.420 (12)	0.478 (12)
200	0.231 (13)	0.219 (13)	0.259 (13)	0.203 (13)	0.231 (13)
R_1/h	ω_1 and (n_{1cr}) ($j = 1,2$)				
50	0.254 (4)	0.252 (4)	0.257 (4)	0.252 (4)	0.254 (4)
100	0.221 (4)	0.220 (4)	0.222 (4)	0.220 (4)	0.221 (4)
150	0.214 (4)	0.214 (4)	0.215 (4)	0.214 (4)	0.214 (4)
200	0.203 (11)	0.200 (11)	0.206 (11)	0.200 (11)	0.203 (11)
R_1/h	ω_1 and (n_{1cr}) ($j = 1$)				
50	0.254 (4)	0.252 (4)	0.266 (4)	0.242 (4)	0.254 (4)
100	0.221 (4)	0.220 (4)	0.230 (4)	0.212 (4)	0.221 (4)
150	0.214 (4)	0.214 (4)	0.223 (4)	0.206 (4)	0.214 (4)
200	0.203 (11)	0.200 (11)	0.214 (11)	0.192 (11)	0.203 (11)
R_1/h	ω_1 and (n_{1cr}) ($j = 2$)				
50	0.254 (4)	0.254 (4)	0.245 (4)	0.265 (4)	0.254 (4)
100	0.221 (4)	0.221 (4)	0.213 (4)	0.230 (4)	0.221 (4)
150	0.214 (4)	0.214 (4)	0.207 (4)	0.223 (4)	0.214 (4)
200	0.203 (11)	0.203 (11)	0.196 (11)	0.211 (11)	0.203 (11)

$\varphi_j(\bar{\zeta}) = -e^{-0.1|\bar{\zeta}|}\cos(\upsilon\bar{\zeta})$ $(j = 1, 2)$ and $\mu = 0.90$ is considered. For $0 \leqslant \upsilon \leqslant 11.6$, the lowest frequency parameter is higher than that for the homogeneous case and takes its maximum value for $\upsilon = 4$ as (0.253), whereas, for $11.6 < \upsilon \leqslant 18$, it takes lower values and is a minimum for $\upsilon = 14.5$, taking the value (0.249). For all values of υ the effect of the variation of Young's moduli and density on the lowest frequency parameter is very little (Fig. 4).

In Fig. 5 are also seen three curves pertaining to the lowest frequency parameter for three different forms of the density variation functions. In curve 1, the case of $\varphi_2(\bar{\zeta}) = e^{-0.1|\bar{\zeta}|}\cos(\upsilon\bar{\zeta})$ and $\mu = 0.90$ is considered. For $0 \leqslant \upsilon \leqslant 6.3$, the frequency parameter is lower than that for the homogeneous case and takes its minimum value for $\upsilon = 0$ as (0.108). For $6 < \upsilon \leqslant 12$, it is higher than the value for the homogeneous case and takes its maximum value for $\upsilon = 10$ as (0.274). Curve 2 corresponds to the homogeneous case and the pertinent value of the lowest frequency parameter is 0.251. In curve 3, the case of $\varphi_2(\bar{\zeta}) = -e^{-0.1|\bar{\zeta}|}\cos(\upsilon\bar{\zeta})$ and $\mu = 0.90$ is considered. For $0 \leqslant \upsilon \leqslant 6.3$, the lowest frequency parameter is higher than that for the homogeneous case and takes its maximum value for $\upsilon = 0$ as (0.718), whereas, for $6.3 < \upsilon \leqslant 12$, it takes lower values and is a minimum for $\upsilon = 8$, taking the value (0.233). For $\upsilon > 4$, the effect of the variation of the density on the lowest frequency parameter is very little (Fig. 5).

In Fig. 6, three curves pertaining the lowest frequency parameter for three different forms of Young's moduli variation functions are drawn. Curve 1 is for the case of $\varphi_2(\bar{\zeta}) = e^{-0.1|\bar{\zeta}|}\cos(\upsilon\bar{\zeta})$ and $\mu = 0.90$. The lowest frequency parameter is higher than that for the homogeneous case if $0 \leqslant \upsilon \leqslant 6.1$ and takes its maximum value for $\upsilon = 0$ as (0.344). If $6.1 < \upsilon \leqslant 12$, it is lower than the value for the homogeneous case and takes its minimum value for $\upsilon = 8$ as (0.200). Curve 2 is for the homogeneous case and the pertinent value of the lowest

Table 10
Variations of the values of P_{1cr}, ω_1 and corresponding wavenumbers for homogeneous and non-homogeneous orthotropic conical shells versus $E_{0S}/E_{0\theta}$ ($L/R_1 = 3$; $\gamma = 45^\circ$, $R_1/h = 200$; $\mu = 0.9$)

$\varphi_j(\bar{\zeta})$	Hom.	$\pm\bar{\zeta}$	$\bar{\zeta}^2$	$-\bar{\zeta}^2$	(+exp)	(−exp)
$E_{0S}/E_{0\theta}$	$P_{1cr} \times 10^6$ and (n_{cr}) ($j = 1$)					
5	0.155 (11)	0.147 (11)	0.173 (11)	0.136 (11)	0.285 (11)	0.025 (11)
10	0.186 (12)	0.176 (12)	0.209 (12)	0.164 (12)	0.343 (12)	0.030 (12)
15	0.208 (12)	0.198 (13)	0.233 (12)	0.184 (13)	0.384 (12)	0.033 (12)
20	0.225 (13)	0.213 (13)	0.252 (13)	0.197 (13)	0.413 (13)	0.036 (13)
25	0.238 (13)	0.226 (13)	0.267 (13)	0.210 (13)	0.439 (13)	0.038 (13)
$E_{0S}/E_{0\theta}$	ω_1 and (n_{1cr}) ($j = 1,2$)					
5	0.143 (9)	0.140 (9)	0.144 (9)	0.140 (9)	0.142 (9)	0.148 (9)
10	0.123 (3)	0.123 (3)	0.123 (3)	0.123 (3)	0.123 (3)	0.124 (3)
15	0.185 (10)	0.183 (10)	0.187 (10)	0.183 (10)	0.185 (10)	0.192 (10)
20	0.199 (11)	0.195 (11)	0.201 (10)	0.195 (11)	0.198 (11)	0.206 (10)
25	0.192 (4)	0.192 (4)	0.193 (4)	0.192 (4)	0.192 (4)	0.193 (4)
$E_{0S}/E_{0\theta}$	ω_1 and (n_{1cr}) ($j = 1$)					
5	0.143 (9)	0.140 (9)	0.150 (9)	0.135 (9)	0.194 (9)	0.055 (9)
10	0.123 (3)	0.123 (3)	0.128 (3)	0.118 (3)	0.168 (3)	0.046 (3)
15	0.185 (10)	0.183 (10)	0.194 (10)	0.176 (10)	0.252 (10)	0.072 (10)
20	0.199 (11)	0.195 (11)	0.209 (10)	0.188 (11)	0.270 (11)	0.077 (10)
25	0.192 (4)	0.192 (4)	0.200 (4)	0.185 (4)	0.262 (4)	0.072 (4)
$E_{0S}/E_{0\theta}$	ω_1 and (n_{1cr}) ($j = 2$)					
5	0.143 (9)	0.143 (9)	0.137 (9)	0.148 (9)	0.104 (9)	0.381 (9)
10	0.123 (3)	0.123 (3)	0.119 (3)	0.128 (3)	0.090 (3)	0.329 (3)
15	0.185 (10)	0.185 (10)	0.179 (10)	0.193 (10)	0.136 (10)	0.496 (10)
20	0.199 (11)	0.199 (11)	0.192 (11)	0.207 (11)	0.146 (11)	0.531 (11)
25	0.192 (4)	0.192 (4)	0.186 (4)	0.200 (4)	0.141 (4)	0.515 (4)

frequency parameter is 0.251. Curve 3 represents the case of $\varphi_2(\bar{\zeta}) = -e^{-0.1|\bar{\zeta}|}\cos(\upsilon\bar{\zeta})$ and $\mu = 0.90$. For $0 \leqslant \upsilon \leqslant 6.1$, the lowest frequency parameter is lower than that for the homogeneous case and takes its minimum value for $\upsilon = 0$ as (0.088), whereas, for $6.1 < \upsilon \leqslant 12$, it takes higher values and is a maximum for $\upsilon = 8$, taking the value (0.272). For $\upsilon > 5$, the effect of the variation of Young's moduli on the lowest frequency parameter is very little (Fig. 6).

It is noted that the exponential function is taken into account as $\exp = \pm e^{-0.1|\bar{\zeta}|}\cos(0.7\bar{\zeta})$, in the following tables and figures.

In Table 7, variations of the values of critical dimensionless hydrostatic pressure P_{1cr} and corresponding circumferential wavenumber n_{cr} for the homogeneous ($\mu = 0$) and the non-homogeneous ($j = 1$) orthotropic truncated conical shells with different non-homogeneity cases, with respect to L/R_1, i.e., for short, medium, long and very long conical shells are presented. As the ratio L/R_1 increases, the values of dimensionless hydrostatic buckling pressure and corresponding wavenumber decrease for the homogeneous and the non-homogeneous orthotropic truncated conical shells with all the non-homogeneity cases. When the non-homogeneous orthotropic truncated conical shell is compared with the homogeneous orthotropic truncated conical shell for three length ratios (short, medium and long); the percent changes in the values of P_{1cr} are 5.3%; −12.5%, 0.1% and −84%, for the positive non-homogeneity cases as linear, parabolic, cubic and exponential, respectively. The percent changes in the values of P_{1cr} are 5.3%; 12.5%, 0.1% and 84%, for the negative non-homogeneity cases as linear, parabolic, cubic and exponential, respectively. The effect of the non-homogeneity on the values of P_{1cr} and n_{cr} is insignificant for very long conical shells, i.e., for $L/R_1 > 10$.

In Table 8, variations of the values of lowest frequency parameter ω_1 and corresponding circumferential wavenumber $n_{1\mathrm{cr}}$ for homogeneous ($\mu = 0$) and non-homogeneous ($j = 1$, $j = 2$ and $j = 1{,}2$) orthotropic truncated conical shells, with respect to L/R_1 are presented. For the homogeneous and non-homogeneous cases, the values of ω_1 and $n_{1\mathrm{cr}}$ are changing depending on variations of the ratio L/R_1; As the ratio $0.25 \leqslant L/R_1 \leqslant 6$ and $L/R_1 > 7$, value of lowest the frequency parameter ω_1 decreases, whereas it increases for $6 \leqslant L/R_1 \leqslant 7$. As the ratio $0.25 \leqslant L/R_1 < 2$, the value of $n_{1\mathrm{cr}}$ increases, whereas, it decreases for $3 \leqslant L/R_1 \leqslant 6$ and $6 < L/R_1 \leqslant 10$. The effect of the non-homogeneity on the values of lowest frequency parameter ω_1 and the corresponding circumferential wavenumber $n_{1\mathrm{cr}}$ is little, in the all cases.

When the density is kept constant $\left(\varphi_2(\bar{\zeta}) = 0\right)$ and only Young's moduli are changed, the higher effects on the lowest frequency parameter are −4.5%, 5.2%, −35% and 62%, for $\varphi_1(\bar{\zeta}) = \bar{\zeta}^2$, $\varphi_1(\bar{\zeta}) = -\bar{\zeta}^2$, $\varphi_1(\bar{\zeta}) = \mathrm{e}^{-0.1|\bar{\zeta}|}\cos(0.7\bar{\zeta})$ and $\varphi_1(\bar{\zeta}) = -\mathrm{e}^{-0.1|\bar{\zeta}|}\cos(0.7\bar{\zeta})$, respectively. When only the density varies through the thickness direction and Young's moduli are kept constant $\left(\varphi_1(\bar{\zeta}) = 0\right)$, the higher effects on the lowest frequency parameter are 3.55%, −4%, 27% and −167%, for $\varphi_2(\bar{\zeta}) = \bar{\zeta}^2$, $\varphi_2(\bar{\zeta}) = -\bar{\zeta}^2$, $\varphi_2(\bar{\zeta}) = \mathrm{e}^{-0.1|\bar{\zeta}|}\cos(0.7\bar{\zeta})$ and $\varphi_2(\bar{\zeta}) = -\mathrm{e}^{-0.1|\bar{\zeta}|}\cos(0.7\bar{\zeta})$, respectively. When Young's moduli vary together with the density in the thickness direction, the higher effects on frequency parameter are low as 1% and 3%, for $\varphi_j(\bar{\zeta}) = \pm\bar{\zeta}^2$ and $\varphi_j(\bar{\zeta}) = -\mathrm{e}^{-0.1|\bar{\zeta}|}\cos(0.7\bar{\zeta}) \quad (j = 1, 2)$, respectively (Table 8).

The values of the lowest or the critical dimensionless hydrostatic pressure of the homogeneous and the non-homogeneous orthotropic truncated conical shells for three length ratios (short, medium and long) with different semi-vertex angles are presented in Figs. 7–9. For conical shells with homogeneous and cubical non-homogeneity cases, since the values of the hydrostatic buckling pressure are approximately same, cubical case is not involved in these figures. From Figs. 7 to 9, it can be seen that as the semi-vertex angle increases, the value of the critical dimensionless hydrostatic pressure decrease for the homogeneous case ($\mu = 0$) and for the non-homogeneous case $\varphi_j(\bar{\zeta}) = \pm\bar{\zeta}^q \quad (q = 1, 2, 3)$ and $\varphi_j(\bar{\zeta}) = \pm\mathrm{e}^{-0.1|\bar{\zeta}|}\cos(0.7\bar{\zeta}) \ (j = 1)$. Furthermore, the variations of $P_{1\mathrm{cr}}$ respect to semi-vertex angle are similar, for three lengths (short, medium and long) of the conical shells.

The computation results of the buckling hydrostatic pressure and the frequency parameter for homogeneous and non-homogeneous orthotropic truncated conical shells with medium length are given in Tables 9–10 and Fig. 10. The values of critical dimensionless hydrostatic pressure, lowest frequency parameter and corresponding wavenumbers for homogeneous and non-homogeneous orthotropic truncated conical shells are listed in Table 9 with respect to R_1/h, as the ratio $L/R_1 = 3$. From Table 9, it is clear that as the R_1/h ratio is increased, the values of $P_{1\mathrm{cr}}$ and ω_1 decreases, whereas, n_{cr} and $n_{1\mathrm{cr}}$ increase for homogeneous and non-homogeneous orthotropic truncated conical shells. As the ratio R_1/h increases, the percentage effects on the critical hydrostatic pressure for the homogeneous and non-homogeneous orthotropic truncated conical shells are nearly equal.

When Young's moduli vary together with the density in the thickness direction, the higher effect on the lowest frequency parameter is 1% for $\varphi_1 = \varphi_2 = \pm\bar{\zeta}^2$. When the only density varies in the thickness direction and Young's moduli are kept constant the higher effect on the lowest frequency parameter is 4% for $\varphi_2 = -\bar{\zeta}^2$. When the density is kept constant and only Young's moduli are changed, the higher effect on the lowest frequency parameter is 5% for $\varphi_1 = \bar{\zeta}^2$. When the variation of Young's moduli and the density are given by power functions, it has been observed that the effect of Young's moduli and the density variation on the critical dimensionless hydrostatic pressure and frequency parameter is most for being parabolic and least for being cubic. Furthermore, when the variation function is negative the conical shell gets more unstable.

Table 10 shows variations of the values of critical dimensionless hydrostatic pressure, lowest frequency parameter and corresponding wavenumbers for homogeneous and non-homogeneous orthotropic truncated conical shells with different non-homogeneity functions versus $E_{0S}/E_{0\theta}$, as the ratio $L/R_1 = 3$. The values of $P_{1\mathrm{cr}}$ and ω_1 are changing depending on variations of $E_{0S}/E_{0\theta}$; as the ratio $E_{0S}/E_{0\theta}$ increases, the values of $P_{1\mathrm{cr}}$ and P_{cr} increase. As $10 \leqslant E_{0S}/E_{0\theta} \leqslant 20$, the values of lowest frequency parameter and corresponding wavenumber increase, however, as $5 \leqslant E_{0S}/E_{0\theta} < 10$ and $20 < E_{0S}/E_{0\theta} \leqslant 25$, these values decrease. Besides, when $E_{0S}/E_{0\theta}$ ratio is increased, the increasing and the decreasing intervals of the values of $P_{1\mathrm{cr}}$ and ω_1 are changed depending on the values of the ratio L/R_1.

The effect of the variation of Young's moduli or density is relevant. When the variation of Young's moduli ($j = 1$) and the density ($j = 2$) are given by power and exponential functions, it is observed that the effect of this non-homogeneity on the critical parameters is relatively more in the case of the exponential function.

Fig. 10 shows variations of the values of critical dimensionless hydrostatic pressure for homogeneous and non-homogeneous orthotropic, truncated conical shells with different non-homogeneity functions versus μ. As μ increases, the value of the critical dimensionless hydrostatic pressure increase.

5. Conclusions

In this study, the free vibration and the stability of non-homogeneous orthotropic truncated conical shells under a hydrostatic pressure are investigated. At first, the basic relations have been obtained for orthotropic truncated conical shells, Young's moduli and density of which vary continuously in the thickness direction. By applying the Galerkin method to the foregoing equations, the buckling pressure and the frequency of vibration are obtained. Finally, carrying out some computations, the effects of the variations of conical shell characteristics, the effects of the non-homogeneity and orthotropy on the critical dimensionless hydrostatic pressure and frequency parameter are found for different mode numbers, when Young's moduli and density vary together and separately.

The numerical results support the following conclusions.

In the case of two different non-homogeneities, where the variation of Young's moduli and the density are represented by power and exponential functions, it is observed that the critical parameters are influenced more from to the exponential function-type non-homogeneity. The separate variation of Young's moduli and density in the thickness direction has a greater influence on lowest frequency parameter when it is compared with their combined effect. Since the non-homogeneity parameter υ is in the first variation zone, non-homogeneity has a considerable influence on the critical parameters.

When $E_{0S}/E_{0\theta}$ ratio is increased, the increasing and the decreasing intervals of the $P_{1\mathrm{cr}}$ and ω_1 values are changed depending on the value of the ratio L/R_1.

As the ratio L/R_1 increases, the value of the dimensionless hydrostatic buckling pressure decreases for the homogeneous and the non-homogeneous orthotropic truncated conical shells with the all non-homogeneity cases.

As the ratio R_1/h increases, the values of $P_{1\mathrm{cr}}$ and ω_1 decrease, whereas, n_{cr} and $n_{1\mathrm{cr}}$ increase for the homogeneous and the non-homogeneous cases. The percentage effects on the critical hydrostatic pressure for the homogeneous and the non-homogeneous orthotropic truncated conical shells are nearly equal.

As the semi-vertex angle γ increases, the values of the dimensionless critical hydrostatic pressure decrease. The effect of the non-homogeneity on the dimensionless critical hydrostatic pressure and dimensionless frequency parameter values is same as in short-, medium- and long-length shells.

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